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# Note

# A Functional Relation and an Acceleration Procedure for Calculating the Voltage Response of Josephson Junctions

## 1. Introduction

The voltage response of a current-driven finite Josephson junction is a time average and a space average. The convergence to the periodic behavior upon which the time average depends can be very slow at low damping. In this note we derive a functional relationship for the voltage response. We then exploit this relationship by means of an extrapolation technique to accelerate the computation of the time average upon which the voltage depends.

Functional relationships which use power series forms of the error in a numerical approximation in order to accelerate or improve a computed solution are well known and are generally called Richardson extrapolation (cf. [1]). The technique introduced here follows this principle in general terms, but is in fact different from it and novel. Computations show the corresponding accelerative technique to be effective. Moreover, it appears to be the only known accelerative technique for the problem treated here.

In the current-driven case and in appropriate units, the jump $\phi(x, t)$ in the argument of the electron wave function (order parameter) across the gap in a Josephson junction is a solution of the following problem for the sine–Gordon equation with damping $\sigma$,

$$\phi_{tt} + \sigma\phi_t - \phi_{xx} + \sin\phi = 0, \qquad 0 \leqslant x \leqslant 1 \tag{1.1}$$

$$\phi_x(0, t) = H, \qquad \phi_x(1, t) = H + I, \qquad t > 0. \tag{1.2}$$

Here $H$ and $I$ are the applied magnetic field and the applied current, respectively, also in appropriate units. The choice of units, while enabling us to normalize the domain of the junction to $x \in [0, 1]$, does not in fact restrict the junction's physical length. The boundary conditions (1.2) correspond to an asymmetrically driven junction and other possibilities exist (cf. [2]).

The initial data $\phi(x, 0)$ and $\phi_t(x, 0)$ are not usually specified since the voltage response is sought and is defined as the following average:

$$V = \lim_{t \to \infty} \frac{1}{t} \int_0^t \int_0^1 \phi_t(\xi, \tau)\, d\xi\, d\tau. \tag{1.3}$$



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Sometimes the space average in this definition is replaced by setting $\xi = 0$, say. The resulting value of V may be seen to be the same (cf. [3]). Although this change would not alter the accelerative method to be introduced, we adhere to the definition (1.3) of V since if provides the most general setting for both the derivation of the functional relation as well as the accelerative computations to be performed. We forego further discussion of the derivation of this problem (1.1)–(1.3) and of the explanation of the variables used since they are well known (cf. [3–5]). The object of this note is to derive a functional relation for V and to exploit this relation to accelerate the computation in (1.3).

In Section 2 we derive the functional relationship (2.6) and introduce the acceleration procedure. In Section 3 we discretize (1.1)–(1.3) and perform computations to illustrate the acceleration. These computations as well as the estimates show the method to be effective.

## 2. Functional Relation and Acceleration Procedure

Calling $\bar{\phi}_t(t) = \int_0^1 \phi_t(\xi, t)\, dt$, we may write (1.3) as

$$V = \lim_{t \to \infty} \frac{1}{t} [\bar{\phi}(t) - \bar{\phi}(0)]. \tag{2.1}$$

Since the limit exists, $\bar{\phi}_t(t)$ must tend, as $t \to \infty$, to a periodic function (actually an almost periodic function) with period $T = 2\pi/V$. Thus $\bar{\phi}(t)$ itself has the form

$$\bar{\phi}(t) = Vt + p(t) + \rho(t), \tag{2.2}$$

where $p(t)$ is a periodic function with period $T = 2\pi/V$ and where $\rho(t)$ is a remainder which vanishes as $t \to \infty$. (Solutions (2.2) are thought to exist for which $T = 2\pi r/V$, where $r$ is some rational number, typically a positive integer. The ensuing derivation is unmodified in this latter case; the corresponding extrapolation formula is not exhibited here since the change is so slight.)

Setting

$$v(t) = \frac{1}{t} (\bar{\phi}(t) - \bar{\phi}(0)) \tag{2.3}$$

and combining with (2.2), we get

$$tv(t) + \bar{\phi}(0) = tV + p(t) + \rho(t). \tag{2.4}$$

Writing this relation with $t$ set equal to $t + T$ and subtracting the result from (2.4) gives

$$(t + T)\, v(t + T) - tv(t) = TV + \rho(t + T) - \rho(t). \tag{2.5}$$

Solving for $V$ $(=2\pi/T)$, we get the following implicit extrapolation formula.

$$V=\frac{2\pi v(t+T)}{2\pi+t(v(t)-v(t+T))+\rho(t+T)-\rho(t)}. \tag{2.6}$$

From this we get our acceleration formula by dropping the $\rho$'s:

$$V_a=\frac{2\pi v(t+T)}{2\pi+t(v(t)-v(t+T))}. \tag{2.7}$$

If we assume that $\rho(t)$ is differentiable and tends to zero like $t^{-1}$, then by the mean value theorem and for some $\theta$, $0<\theta<1$, we have

$$\rho(t+T)-\rho(t)=T\rho'(t+\theta T)=\frac{\text{const}}{t^2}. \tag{2.8}$$

(If $\rho(t)=o(1)$, but not necessarily $O(t^{-1})$, this estimate would change to $o(t^{-1})$. The precise rate of decay of $\rho(t)$ is unknown to us.) Using (2.8) in (2.6) we see that $V-V_a=O(t^{-2})$ (or $o(t^{-1})$ if $\rho(t)=o(1)$). Thus referring to (2.4), we see that the extrapolation formula converges to $V$ with an error $O(t^{-2})$, while the standard method converges like $O(t^{-1})$.

The acceleration formula (2.7) is used as follows. Initial data, $\phi(x,0)$ and $\phi_t(x,0)$ are chosen and the problem (1.1) and (1.2) for $\phi(x,t)$ is solved forward in time. At each time $t$ we compute $v(t)$ from (2.3). Using $v(t)$ as an estimate for $V$, we compute an estimate $T(t)=2\pi/v(t)$ for $T$. Then we advance the computation of $\phi(x,t)$ to the time $t+T(t)$. We use the resulting $v(t+T(t))$ and $v(t)$ itself to compute the accelerated $V_a$ from (2.7). We must emphasize that this need not be the only strategy for using the acceleration formula (2.7). For example, by setting $V_a=V$ and $T=2\pi/V$ in (2.7), the resulting expression may be vieewed as an equation for $V$ to which any effective iteration method might be applied (e.g., Newton's method).

## 3. Computational Verification

To perform computations to test our method, we discretize the entire process. Let $J>0$ be some prescribed integer and introduce the mesh increments $\Delta t=\Delta x=J^{-1}$. The customary discretization of (1.1) and (1.2) produces the following problem for the determination of the $\phi_j^n$, an approximation to $\phi(j\,\Delta x, n\,\Delta t)$, $j=0,\dots,J$, $n=0,1,\dots$.

$$(1+\sigma\,\Delta t)\phi_j^{n+1}=\left(2+\sigma\,\Delta t-\frac{2\,\Delta t^2}{\Delta x^2}\right)\phi_j^n-\phi_j^{n-1}+\frac{\Delta t^2}{\Delta x^2}(\phi_{j+1}^n+\phi_{j-1}^n)-\Delta t^2\sin\phi_j^n, \tag{3.1}$$

$$\phi_1^n-\phi_0^n=H\,\Delta x,\qquad \phi_J^n-\phi_{J-1}^n=(H+I)\,\Delta x,\qquad n=2,3,\dots. \tag{3.2}$$

The initial values, $\phi_j^0$ and $\phi_j^1$, $j=1,\dots,J$, are determined in the standard manner from the initial data. Finally we discretize the $\xi$ integration in (1.3) (equivalently in

TABLE I

| $n$ | $t_n$ | $v(t_n)$ | $[2\pi/(\Delta t v(t_n))]$ | $V_a(t_n)$ |
|---|---|---|---|---|
| | | (a) $\Delta t = 0.05$ | | |
| 500 | 25 | 2.707 | 47 | 3.385 |
| 1000 | 50 | 2.967 | 43 | 3.257 |
| 2000 | 100 | 3.096 | 41 | 3.225 |
| 3000 | 150 | 3.139 | 41 | 3.231 |
| 5000 | 250 | 3.172 | 40 | 3.222 |
| 10000 | 500 | 3.198 | 40 | 3.223 |
| 20000 | 1000 | 3.211 | 40 | 3.224 |
| 30000 | 1500 | 3.215 | 40 | 3.223 |
| | | (b) $\Delta t = 0.025$ | | |
| 500 | 12.5 | 2.135 | 118 | 4.068 |
| 1000 | 25 | 2.644 | 96 | 3.224 |
| 2000 | 50 | 2.893 | 87 | 3.154 |
| 3000 | 75 | 2.974 | 85 | 3.149 |
| 4000 | 100 | 3.016 | 84 | 3.145 |
| 5000 | 125 | 3.04 | 83 | 3.141 |
| 10000 | 250 | 3.09 | 82 | 3.142 |
| 20000 | 500 | 3.114 | 81 | 3.141 |
| 30000 | 750 | 3.123 | 81 | 3.141 |

(2.3)) using Simpson's rule. This is essentially the approach used in [4]. Of course, this discretization of the original problem introduces truncation errors. However, such errors are so well studied that we merely note that with $\Delta x$ sufficiently small, application of the triangle inequality will handle this new source of computational error. The result will be to add to the error estimate a term like $O(\Delta x^2)$ which characterizes the discretization error corresponding to (3.1) and (3.2).

We illustrate the results of our experiments with a typical sample tabulation of the data $n$, $t_n = n\,\Delta t$, $v(t_n)$, $[2\pi/(\Delta t v(t_n))]$,[1] and $V_a(t_n)$. For Table I the following data are used:

$$\sigma = 0.2, \qquad I = 0.8, \qquad H = 4, \qquad \phi(x, 0) = 9, \qquad \phi_t(x, 0) = 0. \tag{3.3}$$

Parts (a) and (b) of the table illustrate the sensitivity of the computations to the choice of $\Delta t = \Delta x$. $\Delta t = 0.05$ in (a) and $\Delta t = 0.025$ in (b). The correct values of the voltage correspond to the finer mesh of part (b) which may be taken as representative of exact values, as determined by standard computational verification. The product $J\,\Delta x = 1$ for both cases (a) and (b) so that the dimensionless junction length is the same for both cases.

[1] The square bracket in this expression, which is also used in the table, signifies taking the integer part.

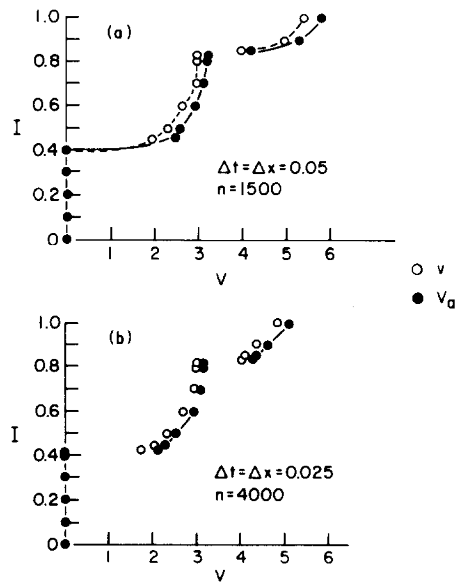


FIGURE 1

As an example notice from part (b) that the unaccelerated value of the voltage $v(t_n) = 3.09 \sim 3.1$ gives the voltage correct to two figures after 10,000 times steps. The accelerative procedure has produced these two figures $V_a(t_n) \sim 3.1$ after about 2000 time steps. $V_a(t_n)$ achieves three figures after 4000 time steps but the unaccelerated value $v(t_n)$ has not yet achieved the third figure even after 30,000 time steps whereupon we abandoned it.

To illustrate exploitation of the method and the pilot calculation of the table, we plot a portion of $V$–$I$ response curve of the Josephson junction in Fig. 1. Here we use the data in (3.3) except that $I$ is varied to produce the response curve. For each $I$, we use $n = 1500$ for part (a) of Fig. 1 and corresponding to part (a) of Table I while we use $n = 4000$ for part (b) of Fig. 1 and corresponding to part (b) of Table I. The plots compare the voltage $v = v(t_n)$ produced by the unaccelerated method with the accelerated voltage $V_a$. Lines connecting some data points in Fig. 1 are drawn only as a convenience. The curves produced have some missing portions. While these must be supplied by more intensive computational effort, we forego producing them since our objective here is to verify the accelerative effect introduced and not to make a systematic study of the junction response. We do however remark that in the units used a resonance (peak) in the response curve is expected for $V = \pi$, in agreement with the computations illustrated here (cf. [4]).[2]

[2] The authors are grateful to the referee for this observation.

## REFERENCES

1. G. DAHLQUIST AND A. BJÖRK, "Numerical Methods," Prentice–Hall, Englewood Cliffs, N.J., 1974.
2. D. L. STUEHM AND C. M. WILMSEN, *J. Appl. Phys.* **48** (1974), 429–433.
3. M. LEVI, F. C. HOPPENSTEADT, AND W. L. MIRANKER, *Quart. Appl. Math.* **36** (1978), 167–198.
4. A. BLACKBURN, J. D. LESLIE, AND H. J. SMITH, *J. Appl. Phys.* **42** (1971), 1047–1054.
5. J. MATISOO, *IEEE Trans. Magnetics* **5** (1969), 848–873.

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W. L. MIRANKER AND K. BANDES*

*Mathematical Sciences Department*
*IBM Thomas J. Watson Research Center, Yorktown Heights*
*New York 10598*

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